

# Boundary Fluctuations and A Reduction Entropy

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The boundary Weyl anomalies live on a codimension-1 boundary,  $\partial\mathcal{M}$ . The entanglement entropy originates from infinite correlations on both sides of a codimension-2 surface,  $\Sigma$ . Motivated to have a further understanding of the boundary effects, we introduce a notion of reduction entropy, which, guided by thermodynamics, is a combination of the boundary effective action and the boundary stress tensor defined by allowing the metric on  $\partial\mathcal{M}$  to fluctuate. We discuss how a reduction might be performed so that the reduction entropy reproduces the entanglement structure.

It is believed that a better understanding of black hole entropy [1, 2] can be achieved by a deeper understanding of the notion of entropy itself, even in flat spacetime [3]. The concept of entanglement entropy (EE), which serves as a measure of the information lost in correlations on either side of a boundary, is widely argued to be a key source of black hole entropy [4, 5]. Assuming the Hilbert space can be factorized into two spatial regions,  $A$  and  $B$ , one defines the EE as

$$S_{\text{EE}} = -\text{tr}(\rho_A \ln \rho_A), \quad (1)$$

where the reduced density matrix,  $\rho_A = \text{tr}_B \rho$ , is obtained by tracing over the degrees of freedom in the complementary region  $B$ ;  $\rho = |\Psi\rangle\langle\Psi|$  is the full density matrix constructed from a pure state.

The standard method to compute the EE is the replica method [6–8], where the path integral is performed on an  $n$ -fold cover of the background geometry with a conical singularity being produced. In this work we focus on the universal EE for  $d = 4$  conformal field theories (CFTs) in flat space with an entangling curved surface  $\Sigma$ . The flat space EE obtained using the conical method [9, 10] is given by

$$S_{\text{EE}} = -\frac{1}{2\pi} \int_{\Sigma} \left( a R_{\Sigma} + c \text{tr} \hat{k}^2 \right) \ln\left(\frac{l}{\delta}\right) + (\text{non-universal}) \quad (2)$$

with

$$R_{\Sigma} = \sum_{a=1}^2 (k_a^2 - \text{tr} k_a^2), \quad \text{tr} \hat{k}^2 = \sum_{a=1}^2 \left( \text{tr} k_a^2 - \frac{1}{2} k_a^2 \right), \quad (3)$$

where  $a = (1, 2)$  represent the coordinates normal to  $\Sigma$ . The non-universal pieces depend on the regularization scheme;  $a$  and  $c$  are central charges. Denoting  $\gamma_{ij}$  as the metric on the codimension-2 manifold  $\Sigma$ , the traceless part of the extrinsic curvature is  $\hat{k}_{ij} \equiv k_{ij} - \frac{k}{2} \gamma_{ij}$ , which transforms covariantly under Weyl transformation;  $R_{\Sigma}$  is the intrinsic Ricci scalar on  $\Sigma$ .

Motivated by the holographic computation of the EE [11], a field theory method was developed in [12], which shows that employing a conformal transformation allows one to map the EE (restricted for a spherical entangling

surface, where only the  $a$ -charge contributes) in CFTs to the ordinary thermodynamical entropy in certain curved spaces. This approach however generates a subtle issue related to the boundary effects; these authors found a mismatch when comparing the thermal entropy with the universal EE. The resolution was recently given by [13], emphasizing the importance of boundary terms in the conformal anomaly. An interpretation of [13] is that the universal EE can be viewed as a purely boundary effect:

$$S_{\text{EE,ball}} = -\widetilde{W}[\delta_{\mu\nu}] + (\text{non-universal}), \quad (4)$$

where  $\widetilde{W}[\delta_{\mu\nu}]$  is the  $a$ -type anomaly effective action with a boundary term, whose expression will be given in the next section, evaluated in *flat* space where only the boundary term contributes. A moral of the computation of [13] is that the universal structure of the EE is already dictated by flat geometry, and the conformal mapping seems somehow unnecessary.

The motivation of this work is to generalize (4), going beyond the spherical surface restriction. We would like to see if the complete universal structure of the EE can be re-captured from boundary anomalies, including the  $c$ -charge contribution, directly from the flat space data. Moreover, we wish not to adopt the conformal mapping or the replica method, but simply to rely on the effective action with boundary terms. In other words, we are interested in finding a new way to compute the EE. To achieve this goal, there are two immediate challenges. First, the universal contribution to the EE comes from a codimension-2 surface while the boundary anomalies live on a codimension-1 manifold; and second, the boundary effective action, as we will discuss more later, related to the  $c$ -charge vanishes in the flat limit.

We will overcome the first challenge by adopting a metric near  $\Sigma$ , which allows us to perform a reduction sending configurations from  $\partial\mathcal{M}$  to  $\Sigma$ . We suggest that the second issue can be resolved by allowing the metric on  $\partial\mathcal{M}$  to fluctuate:  $\delta g_{\mu\nu}|_{\partial\mathcal{M}} \neq 0$ . There then exists the boundary stress tensor contribution, even in the flat limit. Guided by thermodynamics, we will introduce a notion of reduction entropy (RE), which is a combination of the boundary effective action and the boundary

stress tensor. Our main result is to show how the RE reproduces the EE structure and therefore conjecture the relation RE=EE might apply more generally. The discussion of the details we shall leave in the main text. Let us start with a brief review on boundary anomalies.

*Boundary Terms of Conformal Anomaly:* Consider  $d = 4$  CFTs embedded in a curved spacetime  $\mathcal{M}$  with a smooth boundary  $\partial\mathcal{M}$ . The theory can be characterized by the Weyl anomaly. The classification based on the Wess-Zumino consistency [14] was presented in [13]. (See [15, 16] for the classification of the bulk anomaly.) Denoting the induced metric as  $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$  with  $n_\mu$  being a unit, outward normal vector to  $\partial\mathcal{M}$ , the anomaly is given by

$$\langle T_\mu^\mu \rangle = \frac{1}{16\pi^2} \left( cW_{\mu\nu\lambda\rho}^2 - aE_4 \right) + \frac{\delta(x_\perp)}{16\pi^2} \left( aE_4^{(\text{bry})} - b_1 \text{tr} \hat{K}^3 - b_2 h^{AB} \hat{K}^{CD} W_{ACBD} \right) \quad (5)$$

where  $E_4$  is the  $d = 4$  Euler density and  $W_{\mu\nu\lambda\rho}$  is the Weyl tensor.  $\delta(x_\perp)$  is a Dirac delta function with support on the boundary; indices  $A, B, C, \dots$  represent boundary coordinates. The Chern-Simons-like boundary term of the Euler characteristic reads [17]

$$E_4^{(\text{bry})} = -4\delta_{DEF}^{ABC} K_A^D \left( \frac{1}{2} R^{EF}_{BC} + \frac{2}{3} K_B^E K_C^F \right), \quad (6)$$

which is used to supplement  $E_4$  to preserve the topological invariance. Boundary  $b$ -type anomalies, with charges  $b_1$  and  $b_2$ , are defined through the traceless part of the extrinsic curvature,  $\hat{K}_{AB} \equiv K_{AB} - \frac{K}{3} h_{AB}$ , which transforms covariantly under Weyl scaling. The  $b_1$ - and  $b_2$ -type were pointed out first in [18] and [19], respectively.

It is convenient to foliate the spacetime with hypersurfaces labelled by  $r$  and adopt the Gaussian normal coordinates. The metric is given by  $ds^2 = dr^2 + h_{AB}(r, x) dx^A dx^B$ . Using the standard Gauss-Codazzi and Ricci relations, we convert bulk variables into boundary variables and write

$$E_4^{(\text{bry})} = 4 \left( \frac{2}{3} \text{tr} K^3 - K \text{tr} K^2 + \frac{1}{3} K^3 \right) + 8K^{AB} \hat{E}_{AB} \quad (7)$$

$$\text{tr} \hat{K}^3 = \text{tr} K^3 - K \text{tr} K^2 + \frac{2}{9} K^3, \quad (8)$$

$$h^{AB} \hat{K}^{CD} W_{ACBD} = \frac{1}{6} K^3 - \frac{5}{6} K \text{tr} K^2 + \frac{1}{2} K^{AB} \partial_r K_{AB} - \frac{1}{6} K \partial_r K + \frac{1}{2} K^{AB} C_{AB}, \quad (9)$$

where we have denoted  $\partial_r = n^\mu \partial_\mu$ ;  $\hat{E}_{AB}$  is the boundary Einstein tensor and  $C_{AB} = \hat{R}_{AB} - \frac{\hat{R}}{3} h_{AB}$  is the trace-free part of the 3-dimensional Ricci tensor. ( $\hat{R}_{AB}/\hat{R}$  denotes the boundary Ricci tensor/scalar.)

It was recently conjectured that the  $b_2$ -charge is related to the bulk  $c$ -charge by  $b_2 = 8c$ . Using the heat kernel method, ref. [20] confirmed this relation for free fields of spin 0, 1/2, and 1; an argument for this relation based on the variational method was given by [21].

A natural question then, which we tentatively answer in the affirmative, is if one can recover the  $c$ -type EE from this  $b_2$  boundary anomaly. Note that the  $b_2$  anomaly (5) vanishes in flat space, while the  $c$  contribution to the EE (2) requires only a curved boundary. On the other hand, so far there is no indication that the  $b_1$  anomaly, which does not vanish in flat space and which depends on boundary conditions, will contribute to the EE. We will find that  $b_1$  contributes to the entropy in our approach and the known universal EE structure is obtained only when excluding this boundary-condition-dependent charge. (See [22, 23] for earlier discussion of the surface term of the Einstein-Hilbert action and black hole entropy.)

*Response from the Boundary:* Let  $W$  be the effective action including boundary terms. The stress tensor in Euclidean space is defined by  $\langle T^{\mu\nu} \rangle \equiv -\frac{2}{\sqrt{g}} \frac{\delta W}{\delta g_{\mu\nu}}$ . We are interested in the anomaly part of the action, denoted as  $\tilde{W}$ . In the dimensional regularization the anomaly effective action for  $d = 4$  CFTs is essentially given by multiplying the anomaly evaluated in  $4 \rightarrow 4 + \epsilon$  dimensions by an overall factor  $\frac{\mu^\epsilon}{\epsilon}$ , where  $\mu$  stands for a mass scale. (To write the effective action in a more precise manner, one introduces additional vierbeins to construct curvature tensors moving away from the physical dimensions; see for instance [24, 25] for related discussion.) The log contribution comes from the expansion  $\frac{\mu^\epsilon}{\epsilon} \rightarrow \frac{1}{\epsilon} + \ln \mu + \mathcal{O}(\epsilon)$ . Hence, we focus on the variation in the physical dimensions. One should distinguish the divergent part of the stress tensor from the finite part; the effective action is a divergent quantity but the anomaly is a finite quantity obtained by tracing the finite part of the stress tensor. The divergent part of the variational response contributes to the universal entropy. We denote  $\langle T^{\mu\nu} \rangle = \frac{\mu^\epsilon}{\epsilon} \langle t^{\mu\nu} \rangle$  where  $\langle t^{\mu\nu} \rangle$  is obtained by varying the integrated anomaly with respect to the metric.

It is useful to adopt the following expression for the EE in  $d = 4$  CFTs in flat space:

$$\mu \frac{\partial S_{\text{EE}}}{\partial \mu} = c' \frac{\text{Area}(\Sigma)}{(l/\delta)^2} - \frac{1}{2\pi} \int_\Sigma \left( aR_\Sigma + c \text{tr} \hat{k}^2 \right), \quad (10)$$

where  $c'$  is a cut-off-dependent constant;  $\text{Area}(\Sigma)$  is the magnitude of the entangling surface's area. In what follows, we take  $\ln \mu \rightarrow \ln(l/\delta)$  where a dimensional scale  $l$  is inserted to have a dimensionless argument. The quantities we will be computing, the integrated anomaly and  $\langle t^{\mu\nu} \rangle$ , correspond to the contributions to the right-hand-side of (10). By integrating over (10) we have, up to the finite piece, that

$$S_{\text{EE}} = -\frac{c'}{2} \frac{\text{Area}(\Sigma)}{(l/\delta)^2} - \frac{1}{2\pi} \int_\Sigma \left( aR_\Sigma + c \text{tr} \hat{k}^2 \right) \ln \left( \frac{l}{\delta} \right) \quad (11)$$

The first term, the area-law of EE, is sometimes dropped in the literature since the coefficient  $c'$  depends on the way one determines the cut-off. However, we would like to emphasize that the  $a$ -charge does not contribute to  $c'$  while  $c$ -charge does contribute to  $c'$ . (See, for instance, eq(4.28-29) in [26] for the discussion.) In general, there

could be non-anomalous contributions to the  $1/r^2$  term as well. Having this in mind, we would like to see if we can also obtain an additional power-law divergence from the  $c$ -charge related anomaly action.

Allowing the boundary metric to fluctuate gives the boundary stress tensor. The Gaussian normal coordinate is adopted after performing the variation. We fix  $\delta g_{nA} = 0$  where  $n$  is the normal-direction. After performing the variation we take the flat limit so that the bulk contributions are removed. The shape of the boundary remains generally curved. Assuming the boundary is smooth and compact, we perform integration by parts along  $\partial\mathcal{M}$  using the covariant derivative compatible with the boundary metric. We denote  $\overset{\circ}{\nabla}_A$  as the boundary covariant derivative and  $\overset{\circ}{\square} = \overset{\circ}{\nabla}_A \overset{\circ}{\nabla}^A$ ;  $D_n = n^\mu D_\mu$  where  $D_\mu$  is the bulk covariant derivative. The computation is straightforward but tedious in details; here we simply state the final results.

For  $a$ -type, we obtain

$$\begin{aligned} & \lim_{g_{\mu\nu} \rightarrow \delta_{\mu\nu}} \frac{1}{\sqrt{h}} \delta \int_{\partial\mathcal{M}} E_4^{(\text{bry})} \\ &= 4 \int_{\partial\mathcal{M}} \left( K^{AB} (K^2 - \text{tr } K^2) + 2K^{AC} (K_C^D K_D^B - K K_C^B) \right. \\ & \quad \left. - \frac{2}{3} h^{AB} (\text{tr } K^3 - \frac{3}{2} K \text{tr } K^2 + \frac{1}{2} K^3) \right) \delta g_{AB} \end{aligned} \quad (12)$$

Note the normal-normal component and the contribution  $\sim \partial_n \delta g_{AB}$  vanish. In curved-space, the contribution  $\sim \partial_n \delta g_{AB}$  gets cancelled by integrating the bulk action by parts. Moreover, we observe that the stress tensor obtained from the boundary variation (12) can be written as  $t_B^A \sim \delta_{BFGH}^{ACDE} K_C^F K_D^G K_E^H$ , which vanishes identically for any  $d = 3$  boundary, because of the 4 totally-antisymmetric indices. This in fact simply reflects the topological nature of the Euler characteristic, even in the flat limit.

For the  $b_2$ -type, we obtain

$$\begin{aligned} & \lim_{g_{\mu\nu} \rightarrow \delta_{\mu\nu}} \frac{1}{\sqrt{h}} \delta \int_{\partial\mathcal{M}} h^{AB} \hat{K}^{CD} W_{ACBD} \\ &= \int_{\partial\mathcal{M}} \left( A^{AB} \delta g_{AB} + B^{AB} D_n \delta g_{AB} - C \delta g_{nn} \right) \\ & \quad - \frac{1}{4} \int_{\partial\mathcal{M}} \left( \text{tr } \hat{K}^2 \partial_n \delta g_{nn} - \hat{K}^{AB} \partial_n (D_n \delta g_{AB}) \right) \end{aligned} \quad (13)$$

with

$$\begin{aligned} A^{AB} &= \frac{1}{6} K^{AB} (K^2 - \text{tr } K^2) + \frac{1}{2} K^{AC} (K_C^D K_D^B - K K_C^B) \\ & \quad + \frac{1}{12} (\overset{\circ}{\square} K^{AB} - h^{AB} \overset{\circ}{\square} K), \end{aligned} \quad (14)$$

$$\begin{aligned} B^{AB} &= \frac{1}{6} h^{AB} (K^2 - \frac{3}{2} \text{tr } K^2) + \frac{1}{2} K_C^A K^{BC} - \frac{5K}{12} K^{AB}, \\ C &= \frac{1}{2} \text{tr } K^3 - \frac{2K}{3} \text{tr } K^2 + \frac{1}{6} K^3 - \frac{1}{6} \overset{\circ}{\square} K. \end{aligned} \quad (15)$$

This result implies there are normal derivatives of the metric variation contributions left over on the boundary, even in the flat limit. This means the approach of [21],

which derives  $b_2 = 8c$  as a consequence of the variational principle, in fact requires fixing some boundary geometry as a boundary condition (BC). In general, the anomaly effective action however does not need a well-posed variational principle and the choice of a BC would depend on the precise theory one is interested in. (Note we are writing an effective action by integrating out all the dynamical fields; the metric left is an external field.) One of our main tasks therefore is to determine what BC is natural for recovering the entanglement structure. We would like to take the viewpoint that the EE boundary is not really a physical barrier, and the BC imposed for an EE-related computation could be different from that in other considerations; we will discuss more about this point after we introduce the reduction entropy in the next section.

Finally, for the  $b_1$ -anomaly we have

$$\begin{aligned} \frac{1}{\sqrt{h}} \delta \int_{\partial\mathcal{M}} \text{tr } \hat{K}^3 &= \int_{\partial\mathcal{M}} \left( X^{AB} \delta g_{AB} + Y \delta g_{nn} \right) \\ & \quad + \int_{\partial\mathcal{M}} Z^{AB} D_n \delta g_{AB}, \end{aligned} \quad (16)$$

with

$$X^{AB} = \frac{h^{AB}}{2} \text{tr } \hat{K}^3, \quad Y = -\frac{3}{2} \text{tr } \hat{K}^3, \quad (17)$$

$$Z^{AB} = \frac{h^{AB}}{3} (K^2 - \frac{3}{2} \text{tr } K^2) + \frac{3}{2} K^{AC} K_C^B - K K^{AB} \quad (18)$$

*Reduction and Entropy:* We aim to relate the structure on  $\partial\mathcal{M}$  to that of  $\Sigma$  through a notion of reduction entropy. Our basic picture (figure-1) is to first thicken  $\Sigma$  by putting a circle with a radius  $r$  around each point on  $\Sigma$ ; the resulting tube-like manifold is referred to as  $\partial\mathcal{M}$  on which the boundary anomalies live. Having the information (the boundary effective action and the boundary stress tensor) localized on  $\partial\mathcal{M}$ , we want to see how these configurations contribute when being projected on  $\Sigma$ .

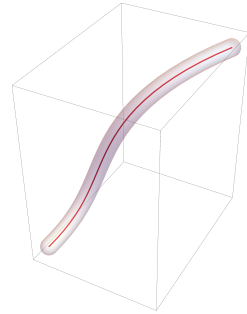


Figure 1: The codimension-2 surface  $\Sigma$  (red line) is thickened to define a codimension-1 boundary  $\partial\mathcal{M}$  (tube), where the boundary metric is allowed to fluctuate. A reduction is performed by sending configurations evaluated on  $\partial\mathcal{M}$  back to  $\Sigma$ .

To perform the reduction, we adopt the metric in the vicinity of  $\Sigma$ . We refer the reader to the appendix B in [27] (see also [10, 28]) for the detailed construction. Performing a Wick rotation to Euclidean space, the metric

moving away from  $\Sigma$  to the second order in the distance is given by

$$ds^2 = \delta_{ab} dx^a dx^b + A_i \epsilon_{ab} x^a dx^b dy^i + \left[ \gamma_{ij} + 2k_{ij}^{(a)} x^a + x^a x^b (\delta_{ab} A_i A_j + k_{im}^{(a)} k_j^{(b)m}) \right] dy^i dy^j + \mathcal{O}(x^3), \quad (19)$$

where  $\{x^a\}_{a=1,2}$  denote the 2-dimensional transverse spaces and  $\{y^i\}_{i=1,2}$  are coordinates parametrizing  $\Sigma$ , which is located at  $x^a = 0$ ;  $\epsilon_{ac}$  is the volume form of the transverse space and  $\gamma_{ij}$  is the corresponding induced metric on the codimension-2 surface. The gauge field  $A_i = \frac{1}{2} \epsilon^{ac} \partial_a g_{ic}|_\Sigma$  is a Kaluza-Klein-like field associated with dimensional reduction on  $\Sigma$ ; this gauge field does not contribute upon reduction in our later computation. (In (19) we have set the background metric to be flat; the metric computed in [27] contains curved-space corrections.) We next make a transformation to polar coordinates by letting  $x^a = r(\cos \theta, \sin \theta)$ . The metric becomes

$$ds^2 = dr^2 + r^2 d\theta^2 + 2r^2 A_i d\theta dy^i + \left[ \gamma_{ij} + 2r \cos \theta k_{ij}^{(1)} + 2r \sin \theta k_{ij}^{(2)} + r^2 \left( A_i A_j + \cos^2 \theta k_{im}^{(1)} k_j^{(1)m} + \sin^2 \theta k_{im}^{(2)} k_j^{(2)m} + \sin(2\theta) k_{im}^{(1)} k_j^{(2)m} \right) \right] dy^i dy^j + \mathcal{O}(x^3). \quad (20)$$

We can use this metric to write the codimension-1 extrinsic curvature,  $K_{AB}$ , as a function of the codimension-2 extrinsic curvature,  $k_{ij}^{(a)}$ :  $K_{AB} \rightarrow K_{AB}(k_{ij}^{(a)}, r, \theta)$ .

We next define a notion of the entropy for this reduction picture. Note the polar coordinates we adopt naturally introduce a temperature defined as the inverse of the periodicity  $2\pi$ . But a consequence of adopting the metric (20) is that we are dealing with non-static configurations. For instance,  $\langle T_\theta^\theta \rangle$  has explicit  $\theta$ -dependence with periodicity  $2\pi$ . We consider a natural extension by integrating the  $\theta$  (time) variable and define the following notion as the “reduction entropy” (RE):

$$S_{\text{RE}} = \lim_{\partial\mathcal{M} \rightarrow \Sigma} \left( -\widetilde{W} + \int_{\mathcal{M}} (\mathcal{E} + \mathcal{P}) \right), \quad (21)$$

where  $\partial\mathcal{M} \rightarrow \Sigma$  stands for a reduction process.  $\widetilde{W}$  is the effective action with boundary terms. The energy density is  $\mathcal{E} \equiv -\langle T_\theta^\theta \rangle$  and the pressure is interpreted here as the normal-normal component of the stress tensor,  $\mathcal{P} = \langle T_r^r \rangle$ . We are largely guided by the thermodynamical entropy in the ensemble maintaining constant temperature and pressure; the corresponding entropy reads  $S = -W + \beta(\langle H \rangle + P\langle V \rangle)$  with  $\beta = 1/T$ . The reduction is performed by integrating  $\theta$  from 0 to  $2\pi$  and taking  $r \rightarrow 0$  to pick up the contribution localized on  $\Sigma$ . In flat space, all bulk contributions are removed. (We find that if instead making an analogy with the canonical ensemble,  $S = -W + \beta\langle H \rangle$ , the c-type EE structure can not be fully recovered and the resulting RE is not Weyl invariant. See the appendix for the discussion. Note that  $\langle T_r^r \rangle$  is non-zero only for  $b_1$ - and  $b_2$ -type actions.)

Let us first consider the simplest case: entropy in  $d = 2$  CFTs. The anomaly effective action with a boundary is

given by

$$\widetilde{W} = -\frac{\mu^\epsilon}{\epsilon} \frac{c_2}{24\pi} \left( \int_{\mathcal{M}} R + 2 \int_{\partial\mathcal{M}} K \right), \quad \epsilon = d - 2, \quad (22)$$

where  $c_2$  stands for the central charge in  $d = 2$ . We focus on the  $d \rightarrow 2$  divergent contribution. The metric variation gives the boundary stress tensor  $t_B^A \sim (K_B^A - h_B^A K) = \delta_{BD}^{AC} K_C^D$ , which vanishes identically for any  $d = 1$  boundary. Thus, the pressure and the energy do not contribute to  $d = 2$  RE. The partition function in flat space is determined by the boundary term. Note here  $\Sigma$  represents two end-points of an entangling interval. We obtain (up to non-universal pieces)

$$S_{\text{RE}} = - \lim_{\partial\mathcal{M} \rightarrow \Sigma} \widetilde{W} = \frac{c_2}{3} \ln\left(\frac{l}{\delta}\right) = S_{\text{EE}}, \quad (23)$$

which is the classic universal EE in  $d = 2$  CFTs [6].

We next turn to the  $d = 4$   $a$ -type contribution. There is no  $a$ -type boundary stress tensor contribution. Thus,

$$\lim_{\partial\mathcal{M} \rightarrow \Sigma} \int_{\mathcal{M}} \langle t_\theta^\theta \rangle^{(a)} = \lim_{\partial\mathcal{M} \rightarrow \Sigma} \int_{\mathcal{M}} \langle t_r^r \rangle^{(a)} = 0. \quad (24)$$

The  $a$ -type RE then solely comes from the partition function, with only the boundary term surviving in flat space. Performing the reduction, we find, up to non-universal terms, that

$$S_{\text{RE}}^{(a)} = - \lim_{\partial\mathcal{M} \rightarrow \Sigma} \widetilde{W}^{(a)} = - \left( \frac{a}{2\pi} \int_{\Sigma} R_{\Sigma} \right) \ln\left(\frac{l}{\delta}\right). \quad (25)$$

This recovers the  $a$ -type EE (2). This expression generalizes the earlier result (4), which is restricted to a spherical surface. The boundary stress tensor was ignored in [13]. However, as shown in the present approach, even if one restores the boundary stress tensor, the result of [13] remains valid.

The more challenging part is to recover also the  $c$ -type EE. Having this  $b_2 = 8c$  relation, we next consider the  $b_2$ -charge contribution to RE. In this case, the RE is given instead only by the boundary stress tensor, since the action simply vanishes in the flat limit. We should now discuss our choice of BC. The standard way to determine the corresponding boundary stress tensor from the result (13) is by imposing certain BCs removing all normal derivatives of the metric variation. However, imposing any Neumann-like BC might not be natural for this entanglement computation because the EE surface (and  $\partial\mathcal{M}$  in the RE picture) should not be viewed as a real boundary. Thus, in this particular computation, we chose *not* to impose any BC or any constraint on the boundary geometry. The resulting  $b_2$ -boundary stress tensor then contains some normal derivatives of the Dirac delta function left over on the boundary.

Let us discuss how these delta functions contribute to RE. We first adopt the following expression in the Gaussian normal coordinates:

$$D_r \delta g_{AB} = \partial_r \delta g_{AB} - K_A^C \delta g_{BC} - K_B^C \delta g_{AC}. \quad (26)$$



The last two terms contribute to the stress tensor in the standard manner. We also have

$$\lim_{g_{\mu\nu} \rightarrow \delta_{\mu\nu}} \partial_r (D_r \delta g_{AB}) = \partial_r \partial_r \delta g_{AB} + K_A^D K_D^C \delta g_{BC} + K_B^D K_D^C \delta g_{AC} - K_A^C \partial_r \delta g_{BC} - K_B^C \partial_r \delta g_{AC}, \quad (27)$$

where we have used that in flat space  $\partial_r K_A^B = -K_A^C K_C^B$ . Including all the contributions, the  $b_2$ -type stress tensor reads

$$\langle t^{AB} \rangle^{(b_2)} = -\frac{b_2}{8\pi^2} (A^{AB} + \Delta_1^{AB} + \Delta_2^{AB}) \delta(r - r') \quad (28)$$

with

$$\Delta_1^{AB} = B^{AB} \partial_r - 2B^{AC} K_C^B, \quad (29)$$

$$\Delta_2^{AB} = \frac{\hat{K}^{AB}}{4} \partial_r^2 - \frac{\hat{K}^{AC} K_C^B}{2} \partial_r + \frac{\hat{K}_C^A}{2} K_D^B K^{CD}, \quad (30)$$

where  $r'$  denotes the location of  $\partial\mathcal{M}$ ; by sending  $r' \rightarrow 0$ , we are performing the reduction back to the location of  $\Sigma$ . We also have

$$\langle t^{rr} \rangle^{(b_2)} = \frac{b_2}{8\pi^2} \left( C + \frac{1}{4} \text{tr} \hat{K}^2 \partial_r \right) \delta(r - r'). \quad (31)$$

Notice in the formula (21) we integrate the density over all space. (The bulk stress tensor vanishes in the flat limit.) Focusing on the normal-coordinate dependence and letting  $f(r)$  be the structure that multiplies the derivative of the delta function, in computing RE we use the property that  $\int dr f(r) \partial_r \delta(r - r') = -\partial_{r'} \int dr f(r) \delta(r - r') = -\partial_{r'} f(r')$  to proceed. Similarly we have  $\int dr f(r) \partial_r^2 \delta(r - r') = \partial_{r'}^2 f(r')$ . Note  $f(r)$  includes the measure factor. We do not perform integration by parts using the normal-derivative on the boundary.

Performing the reduction we obtain, in flat space,

$$\begin{aligned} \mu \frac{\partial}{\partial \mu} S_{\text{RE}}^{(b_2=8c)} &= \lim_{\partial\mathcal{M} \rightarrow \Sigma} \int_{\mathcal{M}} \left( -\langle t_\theta^\theta \rangle + \langle t_r^r \rangle \right)^{(b_2=8c)} \\ &= \frac{c}{3\pi} \frac{\text{Area}(\Sigma)}{r_{\text{cut-off}}^2} - \frac{c}{2\pi} \int_{\Sigma} \text{tr} \hat{k}^2, \end{aligned} \quad (32)$$

where we have adopted the conjectured  $b_2 = 8c$  relation. The first term gives the area-law of the entropy as mentioned in (10). (Such a power-law divergence gets intrinsically cancelled in the  $a$ -type RE.) The second term of (32) reproduces the universal  $c$ -type EE (2). Showing RE=EE from first principles would then provide a proof of  $b_2 = 8c$ .

Finally, we shall also discuss the  $b_1$  boundary anomaly. Note the action does not vanish in flat space. The boundary stress tensor can be read from (16). We have

$$\begin{aligned} \langle t^{AB} \rangle^{(b_1)} &= -\frac{b_1}{8\pi^2} (X^{AB} + Z^{AB} \partial_r - 2Z^{AC} K_C^B) \delta(r - r'), \\ \langle t^{rr} \rangle^{(b_1)} &= -\frac{b_1}{8\pi^2} Y \delta(r - r'). \end{aligned} \quad (33)$$

To be consistent, we again do not impose any BC. Adopting the same method discussed before to perform the reduction, we find that there is no power-law divergence and we obtain

$$S_{\text{RE}}^{(b_1)} = \frac{b_1}{48\pi} \left( \int_{\Sigma} R_{\Sigma} \right) \ln \left( \frac{l}{\delta} \right), \quad (34)$$

up to non-universal terms. We see it contributes like the topological  $a$ -type EE. The result (34) deserves a further understanding. It would be of great interest to see how the  $b_1$ -charge might touch EE. But an interpretation might be that since  $b_1$ -charge is sensitive to BCs imposed on matter fields [20], one might in this sense not regard it as a universal contribution to the entropy. Another possibility, which we do not explore further here, is that  $b_1$  might be related to the  $c$ -charge under certain BCs imposed on matter fields that are suitable in the EE computation; see the appendix for a related viewpoint.

*Conclusion:* In this paper we have tried to establish two main messages: First, the boundary geometry related to the anomaly is rather rich and of great potential importance. The boundary terms are however largely ignored in the literature when constructing a theory containing an entangling surface or a conical singularity; and secondly, it is possible to compute EE directly from flat space, without introducing the  $n$ -fold manifold or performing a conformal mapping to a curved space. Our computation suggests the following identification:

$$S_{\text{RE}} = S_{\text{EE}}, \quad (35)$$

up to non-universal pieces. (The conjecture (35) is formulated by omitting the additional boundary-condition-dependent  $b_1$  contribution to the RE. The expression (35) is then independent of boundary conditions imposed on the matter fields.)

There are however considerable questions and puzzles that are worthy of future study. The most important one perhaps is to have a direct understanding why the universal pieces of the RE and EE should match. A heuristic argument is that the universal structure due to thermal excitations near a codimension-2 surface is indistinguishable from that due to entanglement. Let us here list three further topics for investigation: (1) It would be interesting to restore boundary terms also in the replica approach and see how the boundary terms might interact with the conical singularity. (2) It might be possible to search for new BCs for the boundary metric, different from the one considered here, for RE and compare it with EE. (3) One should better understand why  $\langle t_r^r \rangle$  is important when recovering the EE structure.

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*Appendix:* If we adopt the canonical expression,

$$S_{\text{RE}} = \lim_{\partial\mathcal{M} \rightarrow \Sigma} \left( -\widetilde{W} + \int_{\mathcal{M}} \mathcal{E} \right), \quad (36)$$

we instead find

$$\mu \frac{\partial}{\partial \mu} S_{\text{RE}} = -\frac{b_1}{12\pi} \frac{\text{Area}(\Sigma)}{r_{\text{cut-off}}^2} + \left( \frac{2b_1 + b_2}{16\pi} \int_{\Sigma} \text{tr } \hat{k}^2 - \frac{b_2}{48\pi} \int_{\Sigma} \text{tr } k^2 - \frac{a}{2\pi} \int_{\Sigma} R_{\Sigma} \right). \quad (37)$$

The a-type result is untouched. If matching the above expression with the universal EE one has

$$2b_1 + b_2 = -8c, \quad (38)$$

as a consistency condition. (If  $b_2$  is still  $8c$ , then  $b_1 = -b_2 = -8c$ .) It would be nice to see under what kind of BC does this scenario apply. However, there is an additional  $\sim \int_{\Sigma} \text{tr } k^2$  contribution in (37). We notice such a non-conformal invariant structure appears in [29] as a potential contribution to EE from the scheme-dependent  $\square R$  anomaly. The  $\square R$  term is produced by an  $R^2$  effective action; we emphasize that the corresponding action is *finite*. It reads  $\widetilde{W} \sim \frac{\mu}{\epsilon} (\epsilon R^2)$ . (If one instead takes  $\widetilde{W} \sim \frac{\mu}{\epsilon} R^2$ , the Weyl transformation generates an  $R^2$  which violates the Wess-Zumino consistency condition.) Therefore, the universal part of the RE is free from this  $\square R$  ambiguity. In this sense RE is more robust than EE, and one might instead ask what scheme used in EE can match with RE. It would be interesting to further investigate the potential connection between the boundary anomaly and the  $\square R$  bulk anomaly.

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